Subgradient-Splitting Method for Centralized Multi-Agent Networked System

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Abstract. In this paper, we consider an approximating iterative method for finding a solution of centralized multi-agent network problem by means of the split hierarchical optimization problem. We also discuss convergence results for the sequence generated by the considered method to a solution of the problem.

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1 Introduction

Multi-agent networked systems arise frequently in real world applications and have been vastly interesting in the literature, for example [6–8] and references therein. Let $\mathcal{H}, \mathcal{H}_i$ $(i = 1, \dots, m)$ are finite dimensional Hilbert spaces. In this work, we will consider a multi-agent networked system consisting of a centralized mediator in principal domain $\mathcal{H}$ and a finite number of independent agents i $(i = 1, \dots, m)$ in each individual domains $\mathcal{H}_i$. We assume that each agent i can communicate only to the mediator with an ability operator $A_i : \mathcal{H} \to \mathcal{H}_i$ $(i = 1, \dots, m)$ and it is endowed with a possible decision which can be represented by a fixed point set of operator $S_i : \mathcal{H}_i \to \mathcal{H}_i$ and an i' s cost function $g_i : \mathcal{H}_i \to \mathbb{R}$. We assume that the mediator has its own possible decision which can be represented by a fixed point set of a nonlinear operator $T : \mathcal{H} \to \mathcal{H}$ and take into account global decision. It is worth noting that, in this model, the mediator may only coordinate everything in the system and need not to know any utilities information of agents.

The main target of this centralized multi-agent networked model (in short, **CMNM**) is to find a feasible point $x^* \in \operatorname{Fix}(T) \subset \mathcal{H}$ such that $A_i x^* \in \operatorname{Fix}(S_i) \subset \mathcal{H}_i$, for all $i = 1, \dots, m$, coupling solve

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} g_i(A_i y) \\ \text{subject to} & A_i y \in \operatorname{Fix}(S_i), i = 1, \dots, m. \end{array} \tag{1.1}$$

In order to deal with this problem, we need to recall some useful notions. Let $T : \mathcal{H} \to \mathcal{H}$ be an operator. We denote the set of all fixed points of T by $\operatorname{Fix}(T) := \{x \in \mathcal{H} :$

$x = Tx\}$. An operator T with a nonempty fixed point is called *cutter* if

$$\langle x - Tx, z - Tx \rangle \leq 0,$$

for all $x \in \mathcal{H}$ and all $z \in \mathrm{Fix}(T)$. An operator T is said to be satisfying the *demiclosed principle* if whenever the sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}$ converges weakly to an element $x \in \mathcal{H}$ and the sequence $\{Tx_k - x_k\}_{k\in\mathbb{N}}$ converges strongly to 0, then x is a fixed point of the operator T. For any bounded linear operator A from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$, we denote its adjoint by A^*. We denote the range of A by $\mathrm{Ran}(A) := \{y \in \mathcal{H}_2 : y = Ax, \text{ for some } x \in \mathcal{H}_1\}$. For a subset $D \subset \mathcal{H}_2$, we denote the inverse image of D under A by $A^{-1}(D) := \{x \in \mathcal{H}_1 : Ax \in D\}$.

Let $f : \mathcal{H} \to \mathbb{R}$ and $\bar{x} \in \mathcal{H}$. We remind that an element $x^* \in \mathcal{H}$ satisfies the inequality

$$\langle x^*, x - \bar{x} \rangle + f(\bar{x}) \leq f(x), \qquad \text{for all } x \in \mathcal{H},$$

is called a *subgradient* of f at $\bar{x}$, and the set of all such subgradient is called the *subdifferential* of f at $\bar{x}$; denoted by $\partial f(\bar{x})$. It is well known that if $f : \mathcal{H} \to \mathbb{R}$ is convex and lower semicontinuous, we ensure that $\partial f(\bar{x})$ is a nonempty set, for all $\bar{x} \in \mathcal{H}$, see [10, Theorem 2.4.4].

2 Problem Formulation

For the systematic problem solving, we first assume the following assumption.

Assumption 2.1 *Assume that, for all $i = 1, \ldots, m$, there hold*

(I) $T : \mathcal{H} \to \mathcal{H}, S_i : \mathcal{H}_i \to \mathcal{H}_i$ are cutter operators with fixed points and satisfying the demiclosed principle;

(II) $g_i : \mathcal{H}_i \to \mathbb{R}$ is a convex function;

(III) $A_i : \mathcal{H} \to \mathcal{H}_i$ is a bounded linear operator.

Recall that the product of Hilbert spaces $\mathbf{H} := \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m$ equipped with the addition $\mathbf{x} + \mathbf{y} := (x_1 + y_1, x_2 + y_2, \ldots, x_m + y_m)$, the scalar multiplication $\alpha\mathbf{x} := (\alpha x_1, \alpha x_2, \ldots, \alpha x_m)$ with the inner product defined by

$$\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle_{\mathbf{H}} := \sum_{i=1}^{m} \langle x_i, y_i \rangle_{\mathcal{H}_i},$$

and the norm by

$$\|\mathbf{x}\|_{\mathbf{H}} := \sqrt{\langle\langle \mathbf{x}, \mathbf{x} \rangle\rangle_{\mathbf{H}}},$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbf{H}$, is again a Hilbert space (see [1, Example 2.1]). Let us consider an operator $\mathbf{A} : \mathcal{H} \to \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m$ which is defined by

$$\mathbf{A}(x) := (A_1 x, A_2 x, \ldots, A_m x),$$

for all $x \in \mathcal{H}$ and operator $\mathbf{S} : \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \to \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m$ defined by

$$\mathbf{S}(\mathbf{y}) := (S_1 y_1, S_2 y_2, \ldots, S_m y_m),$$

for all $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m$. Note that the operator $\mathbf{A}$ is a bounded linear operator and $\mathbf{S}$ is cutter with $\mathrm{Fix}(\mathbf{S}) = \mathrm{Fix}(S_1) \times \cdots \times \mathrm{Fix}(S_m)$. Further, defining a function $\mathbf{g} : \mathbf{H} \to \mathbb{R}$ by

$$\mathbf{g}(\mathbf{x}) := \sum_{i=1}^{m} g_i(x_i),$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbf{H}$, we also have that the function $\mathbf{g}$ is a convex function (see [1, Proposition 8.25]). By above setting, we can rewrite **CMNM** as the problem of finding a feasible point $x^* \in \mathrm{Fix}(T) \subset \mathcal{H}$ such that $\mathbf{A}x^* \in \mathrm{Fix}(\mathbf{S}) \subset \mathbf{H}$ solves

$$\begin{array}{ll} \text{minimize} & \mathbf{g}(\mathbf{A}x) \\ \text{subject to} & \mathbf{A}x \in \mathrm{Fix}(\mathbf{S}), \end{array}$$

Notice that this multi-agent network system is a problem of finding a feasible point in a feasible region in a space and its coupling image solves a common decision problem of some corresponding agents in a coupling space. This means that this system is nothing else but the split hierarchical optimization problem which was considered by Nimana and Petrot [9]: let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two finite dimensional Hilbert spaces, $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator, $f : \mathcal{H}_1 \to \mathbb{R}$, $T : \mathcal{H}_1 \to \mathcal{H}_1$ be such that $\mathrm{Fix}(T) \neq \emptyset$, and $g : \mathcal{H}_2 \to \mathbb{R}$, $S : \mathcal{H}_2 \to \mathcal{H}_2$ be such that $A^{-1}(\mathrm{Fix}(S)) \neq \emptyset$. The *split hierarchical optimization problem* (in short, **SHOP**) is to find $x^* \in \mathrm{Fix}(T)$, and such that its image Ax^* solves

$$\begin{array}{ll} \text{minimize} & g(x) \\ \text{subject to} & x \in \mathrm{Ran(A)} \cap \mathrm{Fix}(S), \end{array}$$

Here, we will denote the solution set of **SHOP** by Γ, and the intersection $\mathrm{Fix}(T) \cap A^{-1}(\mathrm{Fix}(S))$ by Ω. And, of course, we will consider the method for approximating a solution of **SHOP** and the convergence properties of such considered method.

3 Convergence Results

We firstly state the core assumption as follows.

Assumption 3.1 *Assume that*

(I) $T : \mathcal{H}_1 \to \mathcal{H}_1, S : \mathcal{H}_2 \to \mathcal{H}_2$ are cutter operators with fixed points and satisfying the demiclosed principle;

(II) $g : \mathcal{H}_2 \to \mathbb{R}$ is a convex function;

(III) $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator.

In order to find a solution of **SHOP**, Nimana and Petrot [9] introduced the the so-called subgradient-splitting method as follows.

Algorithm 3.2 (Subgradient-Splitting Method [9]) *Choose the positive sequences $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{\gamma_k\}_{k\in\mathbb{N}}$ and take arbitrary $x_1 \in \mathcal{H}_1$.*
Step 1: *For a given current iterate $x_k \in \mathcal{H}_1$ $(\forall k \geq 1)$, define $z_k \in \mathcal{H}_2$ $(\forall k \geq 1)$ by*

$$z_k := SAx_k - \alpha_k d_k, \qquad \text{where } d_k \in \partial g(SAx_k).$$

Step 2: *Evaluate* $x_{k+1} \in \mathcal{H}_1$ $(\forall k \geq 1)$ *as*

$$x_{k+1} := T(x_k + \gamma_k A^*(z_k - Ax_k)).$$

Update $k := k+1$ *and go to* **Step 1**.

For simplicity, we will denote $y_k := x_k + \gamma_k A^*(z_k - Ax_k)$ for all $k \geq 1$.

This algorithm 3.2 is a particular situation of the one introduced by the authors in [9] where $f \equiv 0$. One can observe that this algorithm is an integrating ideas of the well known subgradient method and the algorithm for solving the split common fixed point problem [2].

To consider the convergence results for the considered problem, we need an additional key tool. Let C be a nonempty subset of $\mathcal{H}$. We say that a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ is *quasi-Fejér monotone* relative to C, if for all $c \in C$ there exists a sequence $\{\delta_k\}_{k\in\mathbb{N}} \subset [0, +\infty)$ such that $\sum_{k\in\mathbb{N}} \delta_k < +\infty$ and

$$\|x_{k+1} - c\|^2 \leq \|x_k - c\|^2 + \delta_k, \qquad \forall k \geq 1.$$

The following proposition provides some essential properties of a quasi-Fejér monotone sequence, for further information the readers may consult the work of Combettes [3].

Proposition 3.1 *[3] Let $\mathcal{H}$ be a real Hilbert space and $\{x_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ be a quasi-Fejér monotone sequence relative to a nonempty subset $C \subset \mathcal{H}$. Then,*

(i) $\lim_{k\to+\infty} \|x_k - c\|$ *exists for all* $c \in C$.

(ii) If at least one cluster point of $\{x_k\}_{k\in\mathbb{N}}$ lies in C, then $\{x_k\}_{k\in\mathbb{N}}$converges strongly to a point in C.

Now, we will recall some important convergence properties and assumptions used in [9].

Lemma 3.2 *[9, Lemma 3.1] Suppose that Ω is a nonempty set. Then, the following statements hold:*

(i) For all $k \geq 1$ and $q \in \Omega$, we have

$$\begin{aligned} \|x_{k+1} - q\|^2 \ &\leq \ \|x_k - q\|^2 - \gamma_k(2 - \gamma_k\|A\|^2)\|z_k - Ax_k\|^2 + 2\alpha_k\gamma_k\|d_k\|\|z_k - Ax_k\| \\ &\quad +2\alpha_k\gamma_k\left(g(Aq) - g(SAx_k)\right), \end{aligned} \tag{3.1}$$

(ii) For all $k \geq 1$ and $q \in \Omega$, we have

$$\|y_k - q\|^2 \ \leq \ \|x_k - q\|^2 + 2\alpha_k\gamma_k\|d_k\|\|z_k - Ax_k\| \tag{3.2}$$

Assumption 3.3 *The following inclusion holds:*

$$\Gamma \subset \{z \in \Omega : g(Az) \leq g(SAx), \forall x \in \mathcal{H}_1\}.$$

If we let $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex subsets and it holds that $Q \subset \text{Ran}(A)$, then we can set $T := \text{proj}_C$ and $S := \text{proj}_Q$, where proj_C and proj_Q are metric projection onto the set C and Q, respectively, and in this case the assumption 3.3 is satisfied.

Moreover, in this work, we deal with the following control condition.

Condition 3.4 *The sequences* $\{\gamma_k\}_{k\in\mathbb{N}}$ *and* $\{\alpha_k\}_{k\in\mathbb{N}}$ *are satisfying*

(C-1) $0 < \underline{\gamma} := \inf_{k\in\mathbb{N}} \gamma_k \leq \bar{\gamma} := \sup_{k\in\mathbb{N}} \gamma_k < \frac{1}{\|A\|^2}$.

(C-2) $\sum_{k\in\mathbb{N}} \alpha_k = +\infty$, $\lim_{k\to+\infty} \alpha_k = 0$, *and* $\sum_{k\in\mathbb{N}} \alpha_k \|d_k\| < +\infty$.

Now, we present some useful convergence properties.

Lemma 3.3 *Suppose that* $\Gamma \neq \emptyset$, *and Assumption 3.3 and Condition 3.4 hold. If any sequence* $\{x_k\}_{k\in\mathbb{N}}$ *generated by Algorithm 3.2 is bounded then*

(i) $\{x_k\}_{k\in\mathbb{N}}$ *is quasi-Fejér monotone with respect to* Γ, *and* $\lim_{k\to+\infty} \|x_k - q\|$ *exists for all* $q \in \Gamma$.

(ii) $\lim_{k\to+\infty} \|z_k - Ax_k\| = 0$.

(iii) $\lim_{k\to+\infty} \|x_k - x_{k+1}\| = 0$.

(iv) $\lim_{k\to+\infty} \|SAx_k - Ax_k\| = 0$.

(v) $\lim_{k\to+\infty} \|Ty_k - y_k\| = 0$, *and* $\lim_{k\to+\infty} \|x_k - y_k\| = 0$.

Proof. (i) Let $q \in \Gamma$ be given. By Lemma 3.2 and Condition 3.4, we note that

$$\|x_{k+1} - q\|^2 \leq \|x_k - q\|^2 + 2\alpha_k\bar{\gamma}\|d_k\|\|z_k - Ax_k\|, \qquad \forall k \geq 1.$$

Since $\sum_{k\in\mathbb{N}} \alpha_k^2 < +\infty$ and $\sum_{k\in\mathbb{N}} \alpha_k\|d_k\| < +\infty$, we obtain that (i) holds.

(ii) It has been proved in [9, Lemma 3.2 (ii)].

(iii) It is an immediate consequence of the definition of x_{k+1} and (ii).

(iv) Observes that $\|SAx_k - Ax_k\| \leq \|z_k - Ax_k\| + \alpha_k\|d_k\|$ for all $k \geq 1$. Thus, by using (ii) and Condition, we obtain the result in (iv).

(v) Let $q \in \Gamma$. Since $x_{k+1} = Ty_k$, and T is cutter, and using Lemma 3.2 (ii), we have

$$\begin{aligned} \|Ty_k - y_k\|^2 &\leq \|y_k - q\|^2 - \|Ty_k - q\|^2 \\ &\leq \|y_k - q\|^2 - \|x_{k+1} - q\|^2 \\ &\leq \|x_k - q\|^2 - \|x_{k+1} - q\|^2 + 2\bar{\gamma}\alpha_k\|d_k\|\|z_k - Ax_k\|, \qquad \forall k \geq 1, \end{aligned}$$

and hence $\lim_{k\to+\infty} \|Ty_k - y_k\| = 0$, as required. Note that, by using this togethers with (iii), we also have $\lim_{k\to+\infty} \|x_k - y_k\| = 0$. ∎

To obtain the convergence of iterate, we need the following proposition.

Proposition 3.4 (Silverman-Toeplitz' s theorem) *[4,5] Let* $\mathbb{R}^n$ *be a Euclidean space. Let* $a_{lk} \in (0, +\infty)$, *for all* $l \geq 1$ *and* $k = 1, \ldots, l$ *be such that* $\sum_{k=1}^{l} a_{lk} = 1$ *for all* $l \geq 1$ *and* $\lim_{l\to+\infty} a_{lk} = 0$ *for all* $k \geq 1$. *If* $\{u_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ *is a sequence such that* $\lim_{k\to+\infty} u_k = u \in \mathbb{R}^n$, *then* $\lim_{l\to+\infty} \sum_{k=1}^{l} a_{lk}u_k = u$.

By using the Silverman-Toeplitz' s theorem, we can obtain the following result.

Lemma 3.5 *Let* $\mathbb{R}^n$ *be a Euclidean space and* $\{\alpha_k\}_{k\in\mathbb{N}} \subset (0, +\infty)$ *be a sequence such that* $\sum_{k\in\mathbb{N}} \alpha_k = +\infty$. *If* $\{u_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ *is a sequence such that* $\lim_{k\to+\infty} u_k = u \in \mathbb{R}^n$, *then* $\lim_{l\to+\infty} \frac{\sum_{k=1}^{l} \alpha_k u_k}{\sum_{k=1}^{l} \alpha_k} = u$.

Proof. Setting $a_{lk} := \frac{\alpha_k}{\sum_{k=1}^{l}\alpha_k} \in (0,+\infty)$, for all $l \geq 1$, we have $\sum_{k=1}^{l} a_{lk} = 1$ and $\lim_{l\to+\infty} a_{lk} = \lim_{l\to+\infty} \frac{\alpha_k}{\sum_{k=1}^{l}\alpha_k} = 0$. Thus, for all sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ such that $\lim_{k\to+\infty} u_k = u \in \mathbb{R}^n$, we have

$$\lim_{l\to+\infty} \frac{\sum_{k=1}^{l}\alpha_k u_k}{\sum_{k=1}^{l}\alpha_k} = \lim_{l\to+\infty} \sum_{k=1}^{l} \frac{\alpha_k u_k}{\sum_{k=1}^{l}\alpha_k} = \lim_{l\to+\infty} \sum_{k=1}^{l} a_{lk} u_k = u,$$

as desired. ■

Now, we are in position to state the main convergence theorem.

Theorem 3.6 *Suppose that $\Gamma \neq \emptyset$, and Assumption 3.3 and Condition 3.4 hold. If any sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by Algorithm 3.2 is bounded, then $\{x_k\}_{k\in\mathbb{N}}$ converges to an element in Γ.*

Proof. By Proposition 3.1 (ii) and Lemma 3.3 (i), we only need to show that there is at least one cluster point of $\{x_k\}_{k\in\mathbb{N}}$ lies in Γ. Since $\{x_k\}_{k\in\mathbb{N}}$ is bounded, we let $p \in \mathcal{H}_1$ be a cluster point of $\{x_k\}_{k\in\mathbb{N}}$ and a subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ such that $x_{k_j} \to p$ as $j \to +\infty$. It follows that $Ax_{k_j} \to Ap$ and $y_{k_j} \to p$ as $j \to +\infty$, by Lemma 3.3 (v). Also, by employing the demiclosed principles of T and S together with Lemma 3.3 (iv)-(v), we obtain that $p \in \Omega$.

Next, let $q \in \Gamma$ be given. In views of Lemma 3.2 and Assumption 3.3, we note that for every $k \geq 1$

$$2\alpha_k\underline{\gamma}\left(g(SAx_k) - g(Aq)\right) \leq \|x_k - q\|^2 - \|x_{k+1} - q\|^2 + 2\alpha_k\overline{\gamma}D\|z_k - Ax_k\|,$$

where $D := \sup_{k\in\mathbb{N}}\{\|d_k\|\}$. Summing up for $1,\ldots,k_j$, we get

$$2\sum_{i=1}^{k_j} \alpha_i\underline{\gamma}\left(g(SAx_i) - g(Aq)\right) \leq \|x_1 - q\|^2 - \|x_{k_j+1} - q\|^2 + 2\overline{\gamma}D\sum_{i=1}^{k_j}\alpha_i\|z_i - Ax_i\|$$

and then

$$\frac{2\sum_{i=1}^{k_j} \alpha_i\underline{\gamma}\left(g(SAx_i) - g(Aq)\right)}{\sum_{i=1}^{k_j}\alpha_i} \leq \frac{\|x_1 - q\|^2}{\sum_{i=1}^{k_j}\alpha_i} + 2\overline{\gamma}D\frac{\sum_{i=1}^{k_j}\alpha_i\|z_i - Ax_i\|}{\sum_{i=1}^{k_j}\alpha_i}.$$

Since $\lim_{k\to+\infty}\|z_k - Ax_k\| = 0$ and by using Lemma 3.5, we have

$$\lim_{j\to+\infty} \frac{\sum_{i=1}^{k_j}\alpha_i\|z_i - Ax_i\|}{\sum_{i=1}^{k_j}\alpha_i} = 0,$$

and hence, for every $q \in \Gamma$, we have

$$\liminf_{j\to+\infty} \frac{\sum_{i=1}^{k_j}\alpha_i\underline{\gamma}\left(g(SAx_i) - g(Aq)\right)}{\sum_{i=1}^{k_j}\alpha_i} \leq 0.$$

By using the convexity of g, we obtain

$$g\left(\frac{\sum_{i=1}^{k_j}\alpha_i SAx_i}{\sum_{i=1}^{k_j}\alpha_i}\right) \leq \frac{\sum_{i=1}^{k_j}\alpha_i g(SAx_i)}{\sum_{i=1}^{k_j}\alpha_i}, \qquad \forall j \geq 1.$$

Since $SAx_{k_j} \to Ap$ as $j \to +\infty$, then by using Lemma 3.5, we have $\lim_{j\to+\infty} \frac{\sum_{i=1}^{k_j} \alpha_i SAx_i}{\sum_{i=1}^{k_j} \alpha_i} = Ap$. This implies, for every $q \in \Gamma$, that

$$g(Ap) \leq \liminf_{j\to+\infty} g\left(\frac{\sum_{i=1}^{k_j} \alpha_i SAx_i}{\sum_{i=1}^{k_j} \alpha_i}\right) \leq g(Aq).$$

This means $p \in \Gamma$. Therefore, invoking Theorem 3.1 (iii), we conclude that the sequence $\{x_k\}_{k\in\mathbb{N}}$ converges to an element in Γ. ■

Note that the assumption $\sum_{k\in\mathbb{N}} \alpha_k \|d_k\| < +\infty$ is always satisfying whenever g is a constant function. Moreover, if S is the identity operator, then this assumption can be removed. In fact, from Lemma 3.3, we have

$$\|x_{k+1} - q\|^2 \leq \|x_k - q\|^2 + 2\alpha_k\overline{\gamma}\|d_k\|\|z_k - Ax_k\| + \alpha_k^2\|c_k\|^2,$$

for all $q \in \Gamma$ and $k \geq 1$. Since $S = I$, we have $\|z_k - Ax_k\| = \alpha_k\|d_k\|$, which implies that

$$\|x_{k+1} - q\|^2 \leq \|x_k - q\|^2 + 2\alpha_k^2\overline{\gamma}\|d_k\|^2 + \alpha_k^2\|c_k\|^2,$$

for all $q \in \Gamma$ and $k \geq 1$. This means $\{x_k\}_{k\in\mathbb{N}}$ is a quasi-Fejér monotone with respect to Γ.

4 Implication for centralized multi-agent networked system

Accordingly, in order to solve the considered multi-agent network problem, we can rewrite Algorithm 3.2 by doing the suitable substitutions and obtain the following algorithm.

Algorithm 4.1 *Choose the positive sequences $\{\alpha_k\}_{k\in\mathbb{N}}$, $\{\gamma_k\}_{k\in\mathbb{N}}$, and take arbitrary $x_1 \in \mathcal{H}$.*
Step 1: *For a given current iterate $x_k \in \mathcal{H}$ ($\forall k \geq 1$), the mediator inform it to all agents in the system. Each agent i ($i = 1, \ldots, m$) then computes the estimate $z_{k,i} \in \mathcal{H}_i$ as*

$$z_{k,i} := S_iA_ix_k - \alpha_k d_{k,i}, \qquad \text{where } d_{k,i} \in \partial g_i(S_iA_ix_k),$$

and transmits this estimate back to the mediator.
Step 2: *The mediator computes*

$$x_{k+1} := T\left(x_k + \gamma_k \sum_{j=1}^{m} A_j^*(z_{k,j} - A_jx_k)\right).$$

Update $k := k + 1$ and go to **Step 1**.

We now establish a convergence result for **CMNM** which is a consequence of Theorem 3.6.

Theorem 4.1 *Suppose that $\Psi \neq \emptyset$ and the Assumption 2.1 holds. the following conditions hold:*

(i) $0 < \inf_{k\in\mathbb{N}} \gamma_k \leq \sup_{k\in\mathbb{N}} \gamma_k < 1/\sum_{j=1}^m \|A_j\|^2$.

(ii) $\Psi \subset \left\{z \in \mathrm{Fix}(T) \bigcap \cap_{i=1}^m A_i^{-1}(\mathrm{Fix}(S_i)) : g_i(A_i z) \leq g_i(S_i A_i x), \forall x \in \mathcal{H}, i = 1, \ldots, m\right\}$.

(iii) $\sum_{k\in\mathbb{N}} \alpha_k = +\infty$, $\lim_{k\to+\infty} \alpha_k = 0$, *and* $\sum_{k\in\mathbb{N}} \alpha_k \sqrt{\sum_{i=1}^m \|d_i\|_{\mathcal{H}_i}^2} < +\infty$.

If any sequence $\{x_k\}_{k\in\mathbb{N}}$ *generated by Algorithm 4.1 is bounded, then* $\{x_k\}_{k\in\mathbb{N}}$ *converges to an element in* Ψ.

Proof. Firstly, as an above consequence, we define an adjoint operator $\mathbf{A}^* : \mathbf{H} \to \mathcal{H}$ of $\mathbf{A}$ by

$$\mathbf{A}^*(\mathbf{x}) := \sum_{j=1}^m A_j^* x_j,$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbf{H}$. Then, we know that $\partial\mathbf{g}(\mathbf{S}\mathbf{A}x_k) = \partial g_1(S_1 A_1 x_k) \times \cdots \times \partial g_m(S_m A_m x_k)$, see [10, Corollary 2.4.5]. Let us put $\mathbf{d}_k := (d_{k,1}, \ldots, d_{k,m})$ where $d_{k,i} \in \partial g_i(S_i A_i x_k), i = 1, \ldots, m$, for all $k \geq 1$. It follows that $(k \geq 1)$

$$\mathbf{z}_k := (z_{k,1}, \ldots, z_{k,m}) = \mathbf{S}\mathbf{A}x_k - \beta_k \mathbf{d}_k.$$

Also, we observe that, for all $k \geq 1$, it holds

$$\mathbf{A}^*(\mathbf{z}_k - \mathbf{A}x_k) = \sum_{j=1}^m A_j^* \left(z_{k,j} - A_j x_k\right).$$

So, we can rewrite Algorithm 4.1 as

$$\begin{aligned} \mathbf{z}_k &:= \mathbf{S}\mathbf{A}x_k - \alpha_k \mathbf{d}_k; \\ x_{k+1} &:= T\left(x_k + \gamma_k \mathbf{A}^*(\mathbf{z}_k - \mathbf{A}x_k)\right), \end{aligned} \tag{4.1}$$

where $\mathbf{d}_k \in \partial\mathbf{g}(\mathbf{S}\mathbf{A}x_k)$ for all $k \geq 1$. Note that, the form (4.1) is a specialization of Algorithm 3.2. On the other hand, we note that $\mathbf{T}$ and $\mathbf{S}$ satisfy the demiclosed principle. Further, we have

$$\Psi \subset \left\{x \in \mathrm{Fix}(T) \cap \mathbf{A}^{-1}(\mathrm{Fix}(\mathbf{S})) : \mathbf{g}(\mathbf{A}x) \leq \mathbf{g}(\mathbf{S}\mathbf{A}x), \forall x \in \mathcal{H}\right\}.$$

Finally, since $\|\mathbf{A}\|^2 \leq \sum_{j=1}^m \|A_j\|^2$, the result therefore follows from Theorem 3.6. ■

5 Conclusion

This paper discussed the centralized multi-agent network problem by means of the split hierarchical optimization problem introduced by Nimana and Petrot [9]. This introduced model seems a generalization of some multi-agent networked problems. To solve the considerede problem, we employed the algorithm introduced by Nimana and Petrot [9], which we called it by the subgradient-splitting method. We proved the convergence results for this considered problem. It is worth noting that the main result of this work is different from the one in [9] because (1) the convergence result in [9] need the assumption that $\lim_{k\to+\infty} \frac{\|x_{k+1}-x_k\|}{\alpha_k} = 0$, but, in here, it is not necessary, and (2) our convergence result holds true in finite dimensional Hilbert spaces, however, the result in [9] is true even in infinite dimensional Hilbert spaces.

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